ALGEBRA AND ALGEBRAIC THINKING

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1. Introduction

Education has a very important place in today's society. Education is an effective tool for the development and progress of nations and for educating people in the best way possible. A nation's future cannot be shaped without education. Within the education system, mathematics education has an extremely important place in raising people who will work in industry, technology and other areas of daily life, and even academicians who will work in the field of mathematics. The need for mathematics in the field of education in a country and the universality of the language of mathematics are inevitable factors in the progress towards becoming an information society (Yıldız & Uyanık, 2004). Therefore, the importance of mathematics has been taken into account in every period.

Mathematics is one of the oldest sciences in human history. As a science that examines the properties of shapes, numbers, multiplicities and the relations between them, it is divided into many branches of science such as arithmetic, algebra, analysis and geometry. Algebra, as one of these branches of science, which we encounter in all areas of our lives, is a loyal guardian of mathematics (Schoenfeld, 2002). According to some experts, algebra is solving equations, while for others it is calculating. Algebra is a subject that uses a large number of symbols in it (Darojaturrofiah, 2017). Also, algebra provides ideas and symbols to express information about quantitative variables and relationships. In most countries, students use different arithmetic operators such as numbers, arithmetic operations, equal signs, arithmetic expressions and equations before starting to learn basic algebra (Zeller & Barzel, 2010). Because algebra uses symbols to generalize arithmetic (Samo, 2008). Students who begin to learn algebra enter the world of symbols rather than working with numbers. One of the most important steps in learning algebra is to understand the transition of the role of letters in mathematical expressions from unknowns to variables (Ely & Adams, 2012). In other words, it is a transition from concrete numbers to abstract variables. Thus, algebra is of great importance in teaching mathematics. Algebra emerges with many functions in every aspect of our lives. The importance of algebra in our lives is an undeniable fact. Williams (1997) stated that students who do not see learning algebra as a requirement will not be able to understand advanced mathematics courses and that the doors of universities and career jobs will be closed to these students. Also, algebra is at the centre of mathematics learning and affects many areas of mathematics (Irwin & Britt, 2007; Kieran & Drijvers, 2006; Lacampagne, 1995). This importance of algebra in mathematics learning makes learning algebraic thinking a necessity (Usta & Özdemir, 2018). According to Van de Walle, Karp, and Bay-Williams (2014), algebraic thinking is one of the essential elements that dominate all mathematics and make mathematics useful in daily life. Algebraic thinking skills must be developed to be successful in mathematics (Bozkurt, Çırak-Kurt, & Tezcan, 2020). Although algebraic thinking is related to algebra, it has broader meanings than the concept of algebra (Çelik, 2007). Therefore, in this chapter, algebra, algebraic thinking, development levels of algebraic thinking, algebraic thinking styles, development of algebraic thinking and the importance of algebraic thinking will be emphasized.

2. Algebra

Algebra is one of the important learning areas of mathematics (Altun, 2014). Algebra is a branch of mathematics that examines the relationship using numbers and symbols or transforms these relationships into generalized equations (Akkaya, 2006). The foundations of algebra are based on Al-Khwarizmi's work *"El'Kitab'ül-Muhtasar fi Hisab'il Cebri ve'l-Mukabele"* (Summary Book on Algebra and Equation Calculation). Kieran (1992) stated that algebra is a tool that not only represents numbers and numerical data with letters, but also shows number relations and properties, and can also calculate with these symbols (Cited by Akkan et al., 2011). Lacampagne et al. (1995) claimed that only the complete learning of algebraic concepts can open the doors of advanced mathematics and defined algebra as the language of mathematics. Taylor Cox (2003) defined algebra as a more generalized version of arithmetic that contains variables and unknowns to solve the problem. Also, she stated that it enables to make calculations using these symbols. Algebra is based on finding the unknown values with the equations formed by symbolizing them with signs and letters or determining the relations between the unknowns (Argün, Arıkan, Bulut, & Halıcıoğlu, 2014). Dede and Argün (2003) stated that algebra takes on many different functions such as a language, a problem-solving tool, a tool for thought, and a school lesson. They stated that algebra is in every area of life in this respect and it is a necessity for people to learn it. Similarly, Williams and Molina (1997) state that the fact that algebra makes itself felt in all areas of life makes learning it compulsory. Although it is stated as a necessity to learn algebra, it is a lesson that students have difficulty in understanding (Carraher, Schliemann, Brizuela, & Earnest, 2006; Dede & Argün, 2003; Geller & Chart, 2011; Kaput, 1999; Kieran, 1992). This difficulty causes a decrease in students' success in mathematics (Ersoy & Erbaş, 2005). The structure of algebra, students' mental development, readiness levels and deficiencies in teaching algebra can be regarded as obstacles to students' understanding of algebra (Dede & Argün, 2003). Also, algebra is considered more abstract than arithmetic, since it is required to consider all numbers and number sets in algebra (Palabıyık, 2010). Kieran (1992) stated that the main difficulties in learning algebra are the perception of letters and changes in arithmetic and algebraic algorithm, and emphasized that it is necessary to understand what symbols and basic concepts mean to be successful in the field of algebra learning. Gürbüz and Akkan (2008) stated that students had difficulties in transitioning from arithmetic to algebra. They said that the reason for this difficulty was the inadequacy of the student's knowledge of arithmetic operations, the inadequacy of symbolizing and modelling problems, and the inability to use the concept of variables in different situations. Also, they stated that the use of standardized problem types and standardized solution strategies in the learning environment led students to memorization and made the transition from arithmetic to algebra difficult. Akkaya (2006) determined the misconceptions of students about algebra in his study. These are:

- In mathematics, letters have no meaning. For students, letters are used in place of the unknown and have no function.
- Letters are not like numbers. For students, b=d can never be valid because each letter represents a number.
- For students, letters have a digit value and are only seen as numbers.
- Letters are used to abbreviate objects. For example, it is thought that the expression 2e represents 2 eggs.
- Letters are valued according to their position in the alphabet. For example, since the letter d is fifth in the alphabet, its value is thought to be 5.
- The letters are sorted in alphabetical order. For example, if a=3 c=5, students think that b will be 4.
- The “=” sign always gives a result.
- The “+” and “-” signs always give a result.

Welder (2007) defined nine skills as prerequisites for understanding the concept of algebra. These are numerical operations, ratio and proportion, priority in operation, equality, patterns, algebraic symbols, algebraic equations, functions and graphics.

NCTM (2000) emphasized that every student should learn algebra starting from preschool education to the end of high school education. In the book published in the same year, the algebra standards that should be acquired by the students are stated as follows:

- Understanding patterns, relations and functions,
- Being able to analyze and examine mathematical structures using algebraic symbols,
- Understanding numerical relationships using mathematical models,
- Being able to analyze changes from various aspects.

3. Algebraic Thinking

Algebraic thinking, in its most general definition, is the use of variables according to quantitative situations and the capacity to make the relationship between these variables explicit (Driscoll, 1999). In the literature review, it is possible to see different definitions for algebraic thinking. Kieran and Chalouh (1993) put mathematical reasoning at the centre of algebraic thinking. According to them, the basis of algebraic thinking is to understand and use the meaning of mathematical symbols.

Hawker and Cowley (1997) defined this way of thinking as "involves an estimation that requires representation, structuring and generalized thinking of pattern and orders".

In Herbert and Brown's (1997) definition, algebraic thinking is the ability to analyze different situations using mathematical symbols and tools, to express and interpret mathematical information with figures, graphs, tables and equations. Also, algebraic thinking includes proportional reasoning, important ideas, representations, meaning of variables, inductive and deductive reasoning, patterns and functions (Greenes & Findell, 1998). In other words, it is a very comprehensive way of thinking.

Algebraic thinking is the engagement of students with regular roles generalized with mathematical relations and operations, and the establishment of assumptions, arguments and statements in increasingly formal ways through these generalizations (Kaput, 1999; Kaput & Blanton, 1999). In this respect, algebraic thinking skill is one of the high-level mathematical thinking skills.

NCTM (2000) described algebraic thinking as "requires understanding the functions, representing and analyzing mathematical situations and structures in different ways through algebraic symbols, representing and understanding quantitative relationships with mathematical models and analyzing the change in different situations encountered in daily life".

According to Lawrence and Hennessy (2002), algebraic thinking enables us to better interpret the world by translating situations into the mathematical language to explain and predict events in daily life. Also, algebraic thinking develops the ability to think abstractly.

Gülpek (2006) stated that the basic skills of algebraic thinking are generalization, formulation and symbolization. On the other hand, Bağdat (2013) stated that algebraic thinking consists of three basic skills; formulating generalizations, using multiple representations and using algebraic relations and symbols.

Kriegler (2007) stated that algebraic thinking consists of two main components as mathematical thinking tools and informal algebraic relations. Mathematical thinking tools are problem-solving skills (using problem-solving strategies and multiple approaches), representational skills (showing relationships symbolically, visually, numerically, verbally, and transforming different representations) and reasoning (inductive and deductive reasoning) skills. Informal algebraic relations are algebra as abstract arithmetic, algebra as the language of mathematics, and algebra as a tool for working with mathematical modelling and functions.

Çelik (2007) stated that algebraic thinking is not limited to the field of algebra, but is a special form of mathematical thinking. She stated that this thinking consists of three basic skills; using algebraic relations and symbols, using multiple representations (symbolic, graphic, table, etc.) and formulating generalizations. She also emphasized that for people to think algebraically, they should use the meanings of algebraic relations by creating them in their minds and make generalizations by establishing their relationship with real-life situations.

According to Kaya and Keşan (2014), algebraic thinking means establishing relationships between algebraic situations by assigning meanings to symbols as a reflection of mental activities, revealing thoughts through multiple representations and different presentations, describing concrete, semi-concrete and abstract concepts in algebraic relations and reaching conclusions through reasoning.

Algebraic thinking is at every grade level and consists of the following basic topics (Van de Walle, Karp, & Bay-Williams, 2014):

1) The use of patterns that lead to generalizations,

2) Studying the change,

3) Function concept.

The basic idea behind all these topics is that students have a deep understanding of the number system, operations and properties related to operations in achieving success in algebra (Seeley & Schielack, 2008, cited by Van de Walle, 2004). In light of all these explanations, algebraic thinking is among the basic mathematical skills. Some of the senses that affect algebraic thinking are operation sense, structure sense, number sense and symbol sense (Somasundram, 2018). Acar (2019) determined that there is a strong and significant positive correlation between secondary school students' number sense and algebraic thinking levels. NCTM (2000) stated that algebraic thinking skills should be acquired at an early age and that appropriate tools, materials and methods must be used for this. Therefore, the development of students' algebraic thinking skills should be one of the most important goals of mathematics teaching programs.

4. Developmental Levels of Algebraic Thinking

According to the results of the study conducted by Concepts in Secondary Mathematics and Science (CSMS) to determine the level of understanding of algebraic expressions of students aged 13 and 15 in England, students' understanding of algebraic expressions can be examined at four sequential levels (Hart et al., 1998).

<u>Level 1:</u> This level is where questions such as finding the value of a letter as a result of arithmetic operations, concluding a problem by taking letters as objects, or, even though letters are included, concluding an operation without assigning values to these letters, can be solved.

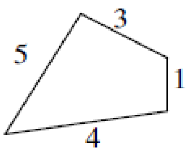
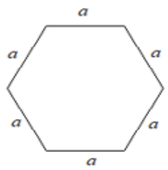
<u>Level 2:</u> This level is the same as the first level in terms of abstraction, but the difference is that the questions are more complex. Students at this level can solve more complex questions.

Level 3: This is the level where letters are perceived and used as unknowns. Since the letters represent an unknown at this level, the student who understands them as an object name cannot reach the correct result.

Level 4: At this level, students can attach meanings to similar but more complex expressions and conclude operations.

For a better understanding of these levels, sample questions about them are given in Table 1.

Table 1. Examples of Questions on Algebraic Thinking Levels (Altun, 2005)

Levels	**Sample Questions**	
Level 1	C 	$3a+2a=?$
Level 2	 $C=?$	if $a=3b+2$, $b=1$ then $a=?$
Level 3	if $e+f=10$ then $d+e+f=?$	$c+d=16$, if $c<d$ then $c=?$
Level 4	Which is it greater; $2n$ or $n+2$? Explain.	Is $a+b+c=a+b+d$ always true? Why?

It is clear that the correct understanding of the symbols used in algebra and the relations between the concepts by the students will contribute to their algebraic thinking. It can be said that it is important to determine the algebraic thinking levels of the students in this respect. Because algebraic thinking contributes to the development of mathematical thinking (Usta & Özdemir, 2018). Therefore, it would be appropriate for mathematics teachers to know the algebraic development levels of students and to provide education accordingly.

5. Algebraic Thinking Ways

Algebraic thinking starts with recognizing numbers and patterns in preschool (NCTM, 2006) and continues with issues related to generalizing and understanding patterns, examining their change, and the concept of function (Van de Walle, Karp & Mr-Williams, 2014). Therefore, algebraic thinking has many properties. Kaput's (1999) five interconnected ways of thinking include:

a) Generalizing arithmetic to algebra: Making generalizations from numbers and arithmetic starts from preschool and continues as long as students continue to learn numbers and operations. While arithmetic is concerned with the numerical value of doing calculations, algebra is concerned with correctly applying mathematical laws and operations to find all solutions. Algebra generalizes arithmetic to find all solutions of equations. Making generalizations about the properties of operations includes noticing patterns that are fundamental to arithmetic and algebra (Russell, Schifter, & Bastable, 2011). In short, by generalizing arithmetic to algebra, it is meant to move the numbers to a higher point.

b) Meaningful use of symbols: Algebra has its own grammar and syntax that can formulate algebraic ideas clearly and effectively (Drijvers et al., 2011). It means that algebra has a language of its own. While students understand this algebraic language, they have difficulties in two subjects that can be considered the basis of algebra. These are the equal sign and the variable concept. Although the equal sign is one of the important symbols in arithmetic, algebra, and all mathematics in general, it is a difficult concept. This is because the equal sign is used to indicate the calculation operation in arithmetic, while in algebra, the equal sign represents the equivalence between two expressions (Carpenter et al., 2003; Molina et al., 2005). Difficulties can be eliminated by reinforcing that the equal sign means sameness in algebra. On the other hand, Mann (2004) suggested teaching the equal sign as a balance between the two sides of the equation. It can be said that teachers can benefit from the scale model while doing this.

Variables can be used as unknown values or varying quantities. Students cannot make sense of how to solve any algebraic equation without knowing the meaning of the variables (Van de Walle et al., 2014). In learning variables as unknown values, students should be familiar with forming mathematical sentences when equations have missing values and a symbol such as an empty square or star represents an unknown value. Also, they need to associate the missing values with the word "variable" (Van de Walle et al., 2014). Students can start learning about variables by using various symbols, letters and symbols to represent unknown quantities (Özden, 2019). In learning variables as changing quantities, the fact that the variables can represent more than one missing value is not understood enough by the students. When there are different variables in a single equation, each variable can represent more than one or even an infinite number (Özden, 2019). It is important to make students aware of this situation.

c) Studying the structures in the number system: Algebraic structure studies are an extension of the analysis of the structure of the number system (Greeno, 1991). Learning the number system requires knowing the properties of real numbers, and this structure can be extended to variables, hence algebra. Moreover, an important way to encourage algebraic thinking is to try to justify assumptions derived from the real number system (Van de Walle et al., 2014). Justifying assumptions at a basic level often requires using examples and adapting them to mathematical thinking to prove that the assumption is true in each example. For example, in an equation using the commutative property (a+b=b+a), it is the main purpose of examining a structure to state that this is true for all numbers (Özden, 2019). The properties of number systems can be created while students work with true/false and open number sentences. However, here, questions such as "Is this true for all numbers?" or "Do you think this is always true? How can we understand?" should be asked to guide students. Students will have to provide reasons to explain their thoughts in this case (Özden, 2019). Therefore, this will lead them to think and interpret more.

d) Examining patterns and functions: Patterns are found in every field of mathematics. Learning how to find patterns, and transfer and extend them to another situation is an important part of algebraic thinking (Van de Walle et al., 2014). Herbert and Brown (1997) stated that the pattern generalization activities carried out by students at an early age are important in forming the basis of algebra and in the development of algebraic thinking. Similarly, Tanışlı and Özdaş (2009) stated that patterns are the building blocks in formulating generalizations, and generalizations are building blocks in algebraic thinking. The relationship learned with the pattern continues with the concept of function with the development of abstract thinking skills in students (Tanışlı & Kabael, 2010). The algebra of functions is mainly concerned with dependent relationships between variables (Drijvers et al., 2011). According to Van de Walle et al. (2014), knowing the function machine, input-output table, concrete, graphical, verbal and symbolic representations and making the transition between these representations easily will enable the development of functional thinking.

e) Mathematical modelling process: Mathematical modelling is defined as the process of starting with real facts and expressing them mathematically (Kaput, 1999). In the context of algebraic thinking, secondary school students should know how to use algebraic models to understand numerical relationships. Converting verbal problems to algebra often involves setting up equations and inequalities and finding the value of one or more variables. So learning how to solve equations and inequalities becomes an essential component of algebra. Mathematical modelling is learned by generalizing arithmetic according to algebra, using symbols in a meaningful way, examining the structure and examining patterns and functions (Özden, 2019). Therefore, it is an issue that needs attention.

6. The Development of Algebraic Thinking

Algebraic thinking is a way of thinking that includes the basic skills necessary for mathematics. This way of thinking includes skills such as understanding variables, reasoning, using representations and explaining the meaning of symbolic representations, working with models, and transforming between representations (Kaf, 2007). Algebraic thinking, which is so important for mathematics, can be developed from an early age. Mathematics educators also emphasize that algebraic thinking should be started in early grades and at an early age (Kieran, 1992). Kieran (2004) listed the things to be considered in the development process of algebraic thinking as follows:

- Students should focus on relationships, not just finding an answer with numerical calculations.
- They should focus not only on the operations themselves and their results but on the reverse of the operations.
- They should focus on both the representation and solution of the problem, not just the solution.
- They should focus not only on numbers but also on what is written.

- They should focus enough on the meaning of the equation sign.

Since students cannot associate their arithmetic knowledge with new knowledge in algebra learning, meaningful learning cannot happen (Çağdaşer, 2008; Gülpek, 2006). Therefore, it was emphasized that arithmetic and algebra should be associated with each other to develop students' algebraic thinking at an early age (Girit & Akyüz, 2016). Lannin, Barker, and Townsend (2006) stated that it is important for students to comprehend the meanings of algebraic symbols and practices in order to enable them to move on to algebraic thinking. Expressing changes and generalizations with graphics, symbols, tables, diagrams or verbally is also seen as important for the development of algebraic thinking (Cañadas, Castro & Castro, 2011). Kieran (1992) emphasized that in order to be successful in learning algebra, it is necessary to understand what symbols and basic concepts mean. Radford (2012) supported the idea of developing algebraic thinking in students at an early age. He mentioned the importance of understanding patterns in early algebraic thinking and emphasized that for the development of algebraic thinking in younger students, key points in algebraic thinking such as equality, problem-solving and generalizing patterns should be understood.

Twohill (2013) stated that the development of algebraic thinking consists of five stages. These stages are pre-formal pattern, informal pattern, formal pattern, generalization and abstract generalization. In the pre-formal pattern stage, students do not identify and cannot recognise patterns. Students who complete this stage are expected to notice the expanding or repeating patterns and express the similarities of the repeating patterns in order to pass to the informal pattern stage. In the next stage, the formal pattern stage, students can explain the pattern verbally and reason about related terms. In the generalization stage, the student can express the rule of the pattern after finding the near term and the far term. In the next stage, abstract generalization, the student can now think algebraically.

7. The Importance of Algebraic Thinking

Algebra allows students to think abstractly and make logical inferences (Stacey & MacGregor, 1997). Lacampagne (1995) said that *"Algebra is the language of mathematics. It opens doors for advanced mathematical subjects if fully learned. If not learned, it closes the doors of the university and technology-based careers..."* (cited by Dede & Argün, 2003). Algebra constitutes a small culture that encompasses a large culture of mathematics.

Although algebraic thinking is related to algebra, it cannot be limited to the subjects of algebra. Algebraic thinking, which helps individuals to solve the problems they face in daily life (Akkan, 2016), is a much wider and more comprehensive field and way of thinking. Algebraic thinking, which is a special form of mathematical thinking, is an initial step in the development of abstract thinking. Besides, algebraic thinking is associated with all areas of mathematics. This way of thinking has a wide and important place in the subject areas of mathematics, which consists of learning areas such as numbers and operations, algebra, geometry and measurement, data processing and probability (MONE, 2013). Algebraic thinking is also used in many fields in other disciplines. Individuals with advanced algebraic thinking are likely to become successful mathematicians, scientists, businesspeople and successful economists (Ahuja, 1998). In other words, it can be said that algebraic thinking also has an effect on students' careers.

Algebraic thinking skills include mental activities to think about, comment on and find solutions to the difficulties that students face not only in mathematics lessons but also in their daily lives. For students to understand algebra and increase their level of algebraic thinking, they should be able to practise what they have learned, make transitions between concepts and demonstrate this with multiple representation values. Every student should learn the necessary levels of algebra from preschool to the end of high school education (NCTM, 2000). This can be achieved by teachers being aware of this situation.

As a result, algebra and algebraic thinking contribute both to the development of mathematical thinking and to understanding the world we live in. In this context, it is very important in terms of mathematics education that students develop their algebraic thinking skills and that teacher candidates and teachers who will put this into practice have the necessary knowledge and skills in this regard.

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